

Viscous Cosmologies with Modified Chaplygin Gas

P. S. Debnath^{#1}, B. C. Paul^{*2}

[#]Physics Department, A. B. N. Seal College,
Coochbehar, Pin : 736 101, India.

^{*}Physics Department, North Bengal University,
Siliguri, Dist. : Darjeeling, Pin : 734 013, West Bengal, India

Abstract —

We present an analysis of the dynamics of cosmological models based on Einstein's general theory of relativity with viscous Modified Chaplygin gas. The evolution of the universe is explored considering the imperfect fluid described by Eckart, truncated Israel and Stewart theories. A number of cosmological models are obtained and found to admit interesting features accommodating the present accelerating phase in addition to early inflationary phase with intermediate deceleration phase of the universe. In the matter dominated universe with viscous Chaplygin gas a new solution is obtained here which is interesting. The stability of the cosmological solutions is also studied.

Keywords — 98.80. Cq, dark energy, viscosity, cosmology, chaplygin gas.

I. INTRODUCTION

Recently a number of cosmological and astronomical observations, namely, supernovae observation [1, 2, 3], when combined with those of the Cosmic microwave background (CMB) radiation, X-rays from the cluster of galaxies [4] and Weak gravitational lensing [5] predict that the present universe might have filled with dark matter (~26.8%) and dark energy (~ 68.3%), the rest (~ 4.9%) are usual baryonic matter. Dark matter (DM) may consist of weakly interacting massive particles (WIMPS) with zero effective pressure particle axions (a particle present in the multiplet of grand unified theories) and neutrinos (light particles present in broken supersymmetric models), but none of these particles could not detected until now. The concept of dark energy (DE) is required to understand the recent acceleration of the universe for which a mysterious entity which is responsible for negative pressure is important to exist. The simplest form of dark energy is the cosmological constant (Λ) (which arises as the result of the combination of quantum field theory and general relativity). However, it becomes essential that the theoretical value of the magnitude of Λ is 60-120 orders of magnitude greater than the observed value [6, 7]. A number of other candidates for the dark energy are self interacting scalar-field dark energy models known as quintessence field [8], phantom [9], K-essence [10, 11, 12], Chaplygin gas are proposed

in order to modify the matter sector of the gravitational action [13].

There exists another way of understanding the observed universe in which dark matter and dark energy are described by a single unified dark energy (UDE) fluid [14]. The Chaplygin gas (CG) is one of the candidates for DE which accommodates accelerated expansion of the universe. The CG plays a dual role at different epoch of the history of the universe: it behaves as a dust like matter in the early time, and like a cosmological constant at late time. This model from the field theory point of view has been investigated [15]. The CG gas emerges as an effective fluid associated with D-branes [16] which may be obtained from the Born-Infeld action [17]. A broad class of UDE models, known as Generalised Chaplygin gas (GCG) is also a candidate. In this framework dark matter and dark energy are just different faces of a single exotic fluid known as Chaplygin gas (CG). It may be pointed out here that Chaplygin gas equation of state [18] was used to describe the lifting force on a wing of an air plane in aerodynamics. When CG cosmology analyzed with observational data namely, SNIa, BAO, CMB and so on [19], it is found that CG is not enough. So an extension of CG model is proposed [20]. Recently, a modified form of Chaplygin gas (MCG) [21, 22, 23, 24, 25] is considered to construct cosmological models. The MCG is more general and it is also found consistent with observations, namely, Gravitational lensing test and Gamma-ray bursts [26, 27, 28].

It is also known that the universe might have emerged to the present state from an inflationary phase in the past. A number of processes might have occurred in the early universe leading to a deviation from perfect fluid assumption e.g. dissipative effect which is to be taken into account in constructing viable cosmological models. Some of the dissipative processes in the early universe may be responsible for emergence of viscosity in the universe. Viscosity may arise due to the decoupling of neutrinos from the radiation era, the decoupling of matter from radiation during the recombination era, creation of superstrings in the quantum era, particle collisions involving gravitons, cosmological quantum particle creation processes and formation of galaxies [29, 30,

31]. As predicted from observation that non negligible dissipative bulks stress may be important at late universe. The possible source of such viscosity may be due to (i) gaseous matter in the framework of relativistic gravity which may give rise to internal self-interaction leading to a negative cosmic bulk pressure [32], (ii) deviation of the non relativistic particle in the substratum from dust. For a non-relativistic substratum cosmic anti-friction may generate a negative fluid bulk pressure which has been noted [33, 34] in the frame work of Einstein gravity. Eckart [35] made the first attempt to describe a relativistic theory of viscosity, however the theories of dissipation in Eckart formulation suffers from shortcoming of causality [36]. The problem arises due to first order nature of the theory, since it considers only first order deviation from equilibrium. It has been shown that the problems of relativistic imperfect fluid may be resolved by including higher order deviation terms in the transport equation [37]. Israel and Stewart [38, 39], and Pavon [40] developed a fully relativistic formulation of the theory taking into account second order deviation terms in the theory, which is termed as "transient" or "extended" irreversible thermodynamics (in short, *EIT*). Using the transport equations obtained from *EIT*, cosmological models are explored in Einstein gravity [41, 42, 43, 44] in addition to CG. Earlier [43] studied viscous Chaplygin gas in a Non-flat universe in the framework of viscosity described by Eckart theory. In the paper modified Friedmann equation due to viscosity is used to determine the time dependent density for a non-flat universe. In this paper we investigate the effect of viscosity on the evolution of a flat universe in the presence of MCG which provide a description of the dark sector of the cosmic medium. In a spatial homogeneous and isotropic universe, the bulk viscous pressure is the only admissible dissipative phenomenon [45] which we consider here.

The plan of this paper is as follows: in sec. 1, we give the gravitational action and set up the relevant field equations. In sec. 2, cosmological solutions are presented. Finally, in sec. 3, we summarize the results obtained.

II. GRAVITATIONAL ACTION & FIELD EQUATIONS

We consider a gravitational action which is given by

$$I = - \int \left[\frac{1}{2} R + L_m \right] \sqrt{-g} d^4x, \quad (1)$$

where R is Ricci scalar curvature, g is the determinant of the four dimensional metric and L_m represents the matter Lagrangian, choosing a unit with $8\pi G = c = 1$.

Variation of the action (1) with respect to $g_{\mu\nu}$ yields

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -T_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu}$ is the energy momentum tensor for matter determined by L_m .

The energy momentum tensor corresponding to the bulk viscous fluid is given by

$$T_{\mu\nu} = (\rho + p + \Pi)u_\mu u_\nu - (p + \Pi)g_{\mu\nu}, \quad (3)$$

where ρ is the energy density and u_μ is the four velocity with normalization condition $u_\mu u_\nu = -1$. Here p is the equilibrium pressure [46] and Π is the bulk viscous pressure.

We consider the homogeneous and isotropic space-time given Friedmann Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (4)$$

where $a(t)$ is the scale factor of the universe. The constant k defined curvature of space, $k = 0, -1, 1$ represents flat, closed, and open spaces respectively. The observations of Cosmic Microwave Background (CMB) anisotropy indicate that the universe is flat and the total energy density is very closed to the critical $\Omega_{tot} \cong 1$ [47]. Hence in proceeding we use the concept of flat universe $k = 0$. The scalar curvature for flat universe ($k = 0$) is

$$R = -6(\dot{H} + 2H^2). \quad (5)$$

Where $H = \frac{\dot{a}}{a}$ is the Hubble parameter, overdot represents derivative with respect to cosmic time t . The trace and (0,0)

components of Eq.(4) are given by field and conservation equations

$$H^2 = \frac{\rho}{3}, \quad 2\dot{H} + 3H^2 = -p - \Pi, \quad (6)$$

$$\dot{\rho} + 3(\rho + p)H = -3\Pi H \quad (7)$$

In *EIT*, the bulk viscous stress Π satisfies a transport equation given by

$$\Pi + \tau\dot{\Pi} = -3\zeta H - \frac{\epsilon}{2} \left[3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T} \right]. \quad (8)$$

Where ζ is the coefficient of bulk viscosity, τ is the relaxation coefficient for transient bulk viscous effects and $T > 0$ is the absolute temperature of the universe. The parameter ϵ takes the value 0 or 1. Here $\epsilon = 0$ represents truncated Israel-Stewart theory (TIS) and $\epsilon = 1$ represents full Israel-Stewart (FIS) causal theory. One recovers the non-causal Eckart theory for $\tau = 0$.

It is assumed that the universe is field with MCG and hence the isotropic pressure p part is described by the equation of state [21-25]

$$p = B\rho - \frac{A}{\rho^\alpha}. \quad (9)$$

Where B ($0 \leq B \leq 1$), α ($0 \leq \alpha \leq 1$) and A ($A > 0$) are dimensional constant. The deceleration parameter (q) is related to H as

$$q = \frac{d}{dt} \left(\frac{1}{H} \right) - 1. \quad (10)$$

The deceleration parameter is negative for accelerating and positive for decelerating phase of the universe.

Using Eqs.(6), (7) and (9) we obtain

$$\Pi = -2\dot{H} - 3 \left[1 + B - \frac{A}{3^{\alpha+1} H^{2\alpha+2}} \right] H^2. \quad (11)$$

The temperature of the universe is obtained through Gibbs integrability condition [48], which is given by

$$n \frac{\partial T}{\partial n} + (\rho + p) \frac{\partial T}{\partial \rho} = T \frac{\partial p}{\partial \rho}. \quad (12)$$

where n is a particle number density, assuming $p = p(\rho, n)$ and $T = T(\rho, n)$. In this paper we assume the temperature and the equilibrium pressure to be barotropic, i.e., $p = p(\rho)$ and $T = T(\rho)$. The adiabatic squared speed of sound $c_s^2 = \frac{\partial p}{\partial \rho} = \frac{dp}{d\rho}$. Eq.

(12) takes the form [49]

$$\frac{1}{T} \frac{dT}{d\rho} = \frac{c_s^2}{\rho + p}. \quad (13)$$

In the absence of particle creation: (i) for a perfect fluid $p = B\rho$ the temperature follows a power-law behaviour which is $T \sim \rho^{\frac{B}{B+1}}$ (ii) in the case of chaplygin gas the temperature varies as $T \sim [(1 + B)\rho^{1+\alpha} - A]^{\frac{B+\alpha+\alpha B}{(1+\alpha)(1+B)}} \rho^{-\alpha}$.

III. COSMOLOGICAL SOLUTIONS

The system of Eqs. (6)-(9) are employed to obtain cosmological solutions. The system of equations is not closed. The coefficients τ and ζ are in general functions of the time (or of the energy density). We assume the following widely accepted *ad-hoc* relations [50]

$$\zeta = \beta \rho^s, \quad \tau = \beta \rho^{s-1} \quad (14)$$

where $\zeta (> 0)$, $\tau (> 0)$, $\beta (> 0)$ and $s (> 0)$ are constant. In the next section we explore cosmologies with Eckart, TIS theory respectively. The FIS theory leads to TIS theory with reduced bulk viscosity at equilibrium (i.e., $\Pi \ll \rho$). The amount of reduction depends on the size of τ related to H . If $\tau H \ll 1$ the reduction is insignificant, i.e., the result of TIS and FIS theory may be comparable.

A. Eckart Theory

The In Eckart theory transport equation take the form

$$\Pi = -\zeta H. \quad (15)$$

Using eq. (11) and (14), the eq. (15) yields,

$$2\dot{H} + 3(1+B)H^2 - \frac{A}{3^\alpha H^{2\alpha}} = \beta 3^{s+1} H^{2s+1}. \quad (16)$$

The above equation is a 1st order autonomous differential equation. A number of cosmological solution may be obtained from eq. (16). We consider the following special cases:

(i) when $s = 0$ and $\alpha = 0$:

In this case the scale factor of the universe yields

$$a(t) = a_0 \left(1 + e^{3k_1(1+B)(t-t_0)} \right)^{\frac{2}{3(1+B)}} \times e^{\left(\frac{\beta t}{2(1+B)} - k_1 t \right)}$$

where a_0, t_0 are the integration constant and $k_1^2 = \frac{A}{3(1+B)} + \frac{\beta^2}{4(1+B)^2}$. It is evident from the cosmological solution that for large (A) and bulk viscosity constant β the rate of expansion is also large. However, when $A=0$, the scale factor of the universe yields

$$a(t) = a_0 \left(1 + e^{\left(\frac{3\beta(t-t_0)}{2} \right)} \right)^{\frac{2}{3(1+B)}}.$$

The solution permits emergent universe scenario [51]. The solution represents a universe which begins with a finite size in the past and grows exponentially ($q < 0$).

(ii) when $s = \frac{1}{2}$: In this special case the Hubble parameter of the universe becomes

$$H = \left(\frac{A(2\alpha+1)(t-t_0)}{2 \times 3^\alpha} \right)^{\frac{1}{2\alpha+1}}$$

with $\beta = \frac{1+B}{\sqrt{3}}$ where t_0 is an integration constant. The scale factor of the universe evolves as $a(t) = a_0 e^{\left(\frac{2\alpha+1}{2\alpha+2} \right) \left(\frac{A(2\alpha+1)}{2 \times 3^\alpha} \right)^{\frac{1}{2\alpha+1}} (t-t_0)^{\frac{2\alpha+2}{2\alpha+1}}}$

In this case the rate of evolution of the scale factor of the universe depends on the parameters of MCG gas EoS. A bounce cosmological model for which the scale factor $a(t)$ is given by exponential form [53] may be obtained for $\alpha = 0$. Here t_0 is the fixed bounce time. The Hubble parameter behaves as $H(t) = \frac{A}{2}(t-t_0)$. For earlier time $t < t_0$, the scale factor decreases, it leads to a contracting universe. At $t = t_0$, the size of the universe is a constant $a(t) = a_0$, which implies a bounce. After the bounce, when $t > t_0$, the scale factor increases and the universe began to expand. The corresponding deceleration parameter is reduced to

$$q = -1 - \left(\frac{2 \times 3^\alpha}{A} \right)^{\frac{1}{2\alpha+1}} \left(\frac{1}{(2\alpha+1)(t-t_0)} \right)^{\frac{2\alpha+2}{2\alpha+1}}.$$

We note the following :

- when $A = 0$ and $B = \beta\sqrt{3} - 1$, it leads to de Sitter type exponential expansion of the scale factor of the universe

$a(t) = a_0 e^{H_0 t}$ even in the presence of matter, where H_0 being a constant.

- when $A = 0$ and $B \neq \beta\sqrt{3} - 1$, a power law expansion of the scale factor of the universe is obtained, which is

$$a(t) = a_0 (t - t_0)^{\frac{2}{3(1+B-\sqrt{3}\beta)}}.$$

It follows accelerated expansion of the universe if $\beta > \frac{3B+1}{3\sqrt{3}}$.

To study the stability of the solution let us define $h = \frac{H}{H_0}$ and rescaling the time as $t^* = H_0 t$ eq.(16) can be written as

$$\dot{h} = f(h) \tag{17}$$

where $f(h) = \frac{A_1}{h^{2\alpha}} + \beta_1 h^{2s+1} - B_1 h^2$, $A_1 = \frac{A}{2 \times 3^\alpha H_0^{2\alpha+2}}$, $\beta_1 = \frac{1}{2} \beta 3^{s+1} H_0^{2s-1}$, $B_1 = \frac{3(B+1)}{2}$ and

(.) represents derivative with respect to time (t^*). To determine the solution of the above eq. (17) we consider an arbitrary initial condition h_0 . As time goes on, the phase point at h_0 moves along the h-axis according to some function $h(t^*)$ on the phase plane \dot{h} vs h . This function is called the trajectory based at h_0 and it represents the solution of the differential equation starting from h_0 . The qualitatively different trajectories of the system is called a phase portrait. The appearance of the phase portrait is controlled by a equilibrium point h^* , determined by $f(h^*) = 0$. The fixed point represent equilibrium or steady solution.

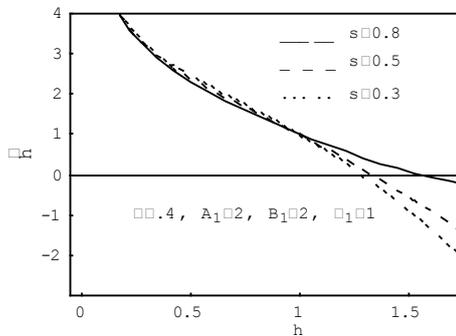


Fig. 1 Variation of \dot{h} vs h for Different Values of s , with $\alpha = 0.4, A_1 = 2, \beta_1 = 1$ and $B_1 = 2$.

Fig. (1) shows how values of the fixed point h^* vary with s for a given set of other parameters. The values of the fixed point are larger for bigger value of s .

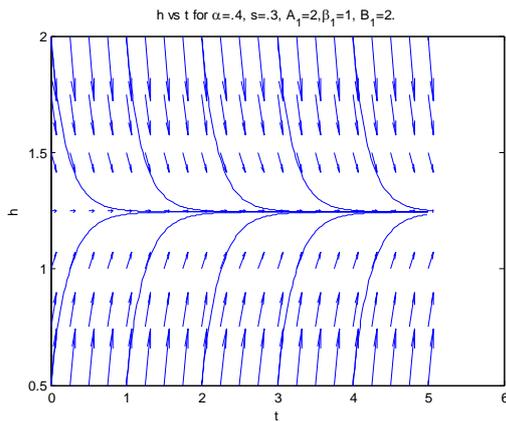


Fig. 2 Directional Plot of h vs t with $s = 0.3, \alpha = 0.4, A_1 = 2, \beta_1 = 1$ and $B_1 = 2$.

Fig. (2) shows that the directional plot of h vs. t which is controlled by the fixed point (h^*). An equilibrium or fixed point is defined to be stable if all sufficient small deviation away from it damp out in time. Conversely in unstable equilibrium, all sufficient small deviation grow in time. The stability of the fixed point can be determined by a local

stability analysis and this is summarized in table 1. We consider the following special case of fixed point and determine their stability by linear stability analysis method [52] as shown in table 1.

Using Eq. (10), Eq. (16) can be rewritten in terms of deceleration parameter as

$$q = \frac{1}{2} + \frac{3B}{2} - \frac{1}{2} \frac{A}{3^\alpha H^{2\alpha+2}} - \frac{1}{2} \beta 3^{s+1} H^{2s-1}. \tag{18}$$

In the above equation, the 3rd and 4th terms in the right hand side arises for Chaplygin gas and bulk viscosity respectively. The combination of bulk viscosity with Chaplygin gas here acts as dark energy of the universe. We note here the following case for:

i) $s = \frac{1}{2}, A = 0, B \neq \beta\sqrt{3} - 1$: The value of the deceleration parameter becomes constant (but $q \neq -1$) i.e., the evolution of the universe is Friedmann-like behaviour,

ii) $s = \frac{1}{2}, A = 0, B = \beta\sqrt{3} - 1$: the value of the deceleration parameter becomes unity with negative sign (i.e., $q = -1$) i.e., the evolution of the universe exhibits de Sitter type expansion even in the presence of matter.

B. Truncated Israel and Stewart Theory (TIS)

In TIS theory transport equation take the form

$$\Pi + \tau \dot{\Pi} = -3\zeta H. \tag{19}$$

Using Eq.(11) and Eq. (14) in TIS transport Eq.(19), we obtain [41]

$$\ddot{H} + b_1(H) H \dot{H} + b_2(H) H^3 = 0. \tag{20}$$

Where $b_1(H) = 3(B + 1) + \frac{A\alpha}{3^\alpha H^{2\alpha+2}} + \frac{1}{\beta 3^{s-1} H^{2s-1}}$, $b_2(H) = \frac{1+B}{2\beta 3^{s-2} H^{2s-1}} - \frac{9}{2} - \frac{A}{2\beta} \times \frac{1}{3^{s+\alpha-1}} \times \frac{1}{H^{2s+2\alpha+1}}$.

The Eq. (20) is the equation of evolution of the universe in the TIS theory.

To obtain Power-law expansion of the scale factor of the universe $a(t) = a_0 t^D$ in the TIS theory, we note the following special case:

(i) If $A=0$: In this case the eq. (20) yields $P_1 + P_2 t^{2s-1} = 0$. $\tag{21}$

Where

$$P_1 = \beta 3^s \left(2 - 3D(1+B) - \frac{9}{2} D^2 \right) \text{ and } P_2 = \left((1+B) \frac{9}{2} - 3 \right) D^{2-2s}.$$

For $s \neq \frac{1}{2}$ both P_1 and P_2 are zero which leads to $\beta = 0$ i.e., no viscosity and hence not interesting. If $s = \frac{1}{2}$ we get $P_1 + P_2 = 0$, i.e., $\left((1+B)3\sqrt{3} - 9\beta \right) D^2 - 2D(3\beta(1+B) + \sqrt{3}) + 4\beta = 0$

$$\tag{22}$$

For simplicity let us consider $\beta = \frac{2\sqrt{3}}{11}$. In this case, a matter dominated universe i.e., $B=0$, the scale factor of the universe evolves as $a(t) = a_0 t^{\frac{17+13}{15}}$. For radiation dominated universe i.e., $B = \frac{1}{3}$, the scale factor of the universe evolves as $a(t) = a_0 t^{\frac{19+\sqrt{153}}{26}}$.

It is evident that both matter and radiation dominated era universe is expanding but during matter dominated regime a new solution $a(t)_+ = a_0 t^2$, is found in the presence of viscosity which corresponds to power law inflation.

For stiff fluid $B=1$ with Chaplygin gas, the scale factor of the universe evolve as $a(t) = a_0 t^{\frac{23+\sqrt{145}}{48}}$. The eq. (20) is the basic equation to study the evolution of the universe in the TIS theory. The solution of the equation leads to a relation for H and \dot{H} [31]. In the standard theory, only one condition for H is required and is usually taken to be $H_0 = \infty$. In the present case, the initial value of H can be finite, even negative. Introducing the quantity $h = \frac{H}{H_0}$ and rescaling the time as $t^* = H_0 t$ the Eq. (20) yields

$$2\beta_1 h^{2s-2} \ddot{h} + \left[1 + \frac{4}{9} \beta_1 B_1 h^{2s-1} + \frac{4}{9} \times \frac{\beta_1 B_1 \alpha h^{2s-1}}{h^{2\alpha+3}} \right] \dot{h} + h^2 \left[B_1 - \beta_1 h^{2s-1} - \frac{A_1}{h^{2\alpha+2}} \right] = 0. \quad (23)$$

Where $A_1 = \frac{A}{2 \times 3^\alpha H_0^{2\alpha+2}}$, $B_1 = \frac{3(1+B)}{2}$, $\beta_1 = \frac{1}{2} \beta H_0^{2s-1} 3^{s+1}$ and H_0 is a constant. An upper dot denote derivation with respect to t^* . The eq. (23) can have no periodic solution on phase path lies in any region as coefficient of \dot{h} is of one sign. Such regions have only positive damping or negative damping. The above second order differential equation governs the evolution of the reduced Hubble parameter h which has two different stationary trivial solutions $h = h_1 = const.$ and $h=0$. The first implies inflationary expansion with a constant rate given by $H_0 h_1$. One can study the behaviour of h near the steady solution analytically. Setting $h = h_1 + \chi$, with $|\chi \ll h_1|$, after linearization eq. (23) yields,

$$\alpha_2 \ddot{\chi} + \alpha_1 \dot{\chi} + \alpha_0 \chi = 0. \quad (24)$$

Where $\alpha_2 = \frac{2}{9} \beta_1 h_1^{2s-2}$, $\alpha_1 = 1 + \frac{4}{9} A_1 \alpha \beta_1 h_1^{2s-2\alpha-3} + \frac{4}{9} \beta_1 B_1 h_1^{2s-1}$ and $\alpha_0 = 2A_1(1+\alpha)h_1^{2\alpha-1} - 2\beta_1(2s-1)h_1^{2s}$.

The solution of the above equation yields

$$\chi(t) = \chi_1 e^{\delta_+ t} + \chi_2 e^{\delta_- t}, \quad (25)$$

where χ_1 and χ_2 are constants which depend on initial condition, while δ_+ and δ_- are the roots

$$\delta(\pm) = \frac{\alpha_1}{\alpha_2} \left[-1 \pm \sqrt{1 - \frac{4\alpha_2\alpha_0}{\alpha_1^2}} \right]. \quad (26)$$

For $\alpha_1 < \sqrt{4\alpha_0\alpha_2}$, eq. (26) gives rise to a damped oscillatory behaviour for h at $h = h_1$. For an oscillatory damped behaviour the following constraint is imposed on A_1 : $3(1+B) > \frac{A_1}{h_1^{2\alpha+2}} >$

$\frac{(2s-1)(1+B)}{2s+2\alpha+1}$ When $A_1 = 0$ and $s < \frac{1}{2}$, one obtains also damped oscillatory behaviour.

So due to the presence of MCG the permitted range of values of s is increased for realizing damped oscillatory behaviour around an exponential inflation. Now if the quantity under the square root of the right hand side of Eq. (26) is positive then one of the roots will be positive, therefore the state with $h = h_1$ will be unstable. We note the following:

(1) When $A_1 = 0$: In this case value of $h_1 = \left(\frac{B_1}{\beta_1}\right)^{\frac{1}{2s-1}}$, the quantity with in the square root of the Eq. (26) will becomes negative if $s < \frac{1}{2}$ [54], and h shows an oscillatory damped behaviour around h_1 with frequency

$$\omega = \frac{9}{2\beta_1 h_1^{2s-2}} \sqrt{\left(1 + \frac{4}{9} B_1^2\right)^2 + \frac{4}{9} (2s-1) B_1^2}. \quad (27)$$

The condition for damped oscillatory is $s < \frac{1}{2} - \frac{9\left(1 + \frac{4}{9} B_1^2\right)^2}{16 B_1^2}$.

(2) When $s = \frac{1}{2}$: In this case $h_1 = \left(\frac{A_1}{B_1 - \beta_1}\right)^{\frac{1}{2\alpha+2}}$, leads to a complex ω when $\frac{8\beta_1(1+\alpha)(B_1 - \beta_1)}{9(1 + \frac{4}{9} B_1 \beta_1 + \frac{4}{9} \alpha \beta_1 (B_1 - \beta_1)^2)} > 1$. An oscillatory damped behaviour around h_1 is permitted with frequency

$$\omega = \frac{9 h_1}{2\beta_1} \sqrt{\omega_0^2 - \frac{8(1+\alpha)(B_1 - \beta_1)}{9}}, \quad (28)$$

where $\omega_0 = 1 + \frac{4B_1\beta_1(1+\alpha)}{9} - \frac{4}{9} \alpha \beta_1^2$. To study the stability analysis of the critical point of the 2nd order autonomous differential equation, the above differential equation may be reduced to a set of two first order differential equation taking \dot{h} as an independent variable which are given by:

$$\begin{aligned} \dot{h} &= y \quad (29) \\ \dot{y} &= P(h, y) = \frac{h^3}{3} + \frac{A_1}{2\beta_1 h^{2\alpha+2s-2}} - \frac{B_1}{2\beta_1 h^{2s-4}} \\ &\quad - \left(\frac{2B_1 h}{9} + \frac{2A_1 \alpha}{9h^{2\alpha+1}} + \frac{1}{2\beta_1 h^{2s-2}} \right) y. \quad (30) \end{aligned}$$

The appearance of the phase portrait is controlled by the fixed or equilibrium point h^* , determined by $P(0, h^*) = 0$ as at the critical point $\dot{y} = y = 0$. Using Taylor's expansion of eq.(30) around critical point can be written as

$$\dot{y} = \frac{\partial P(h, y)}{\partial h} \Big|_{h^*} h + \frac{\partial P(h, y)}{\partial y} \Big|_{h^*} y + Q(x, y). \quad (31)$$

Using linear approximation in the neighbourhood of the equilibrium point, one can write eq. (31) as

$$\dot{y} = b h + c y, \quad (32)$$

where $b = \frac{3}{2} h^{*2} + \frac{A_1}{\beta_1} (1 - s - \alpha) h^{*1-2s-2\alpha}$ and $c = -\frac{2B_1 h^*}{9} + \frac{2A_1 \alpha}{9h^{*2\alpha+1}}$

The stability of the fixed point can be determined [55] by a local stability analysis and this is summarized in table 2.

The result of analysis for nature of fixed points and allowed range of values of the stability conditions are displayed in table 2 for three special cases.

In this case no fixed point found which is centre in the presence of Modified Chaplygin gas.

Non-stationary solutions are permitted as evident from numerical analysis [54,56].

| | | | |
|--------------------------------|-----------------|---------------------------------------|---|
| As = 0, $\alpha = \frac{1}{2}$ | $h_0 = const.,$ | Stable Spiral for $s < \frac{1}{2}$, | $\left(\frac{2B_1}{9} + \frac{1}{2\beta_1}\right)^2 < 2(1-2s)$ |
| | | Saddle | Unstable for $s > \frac{1}{2}$ |
| A = 0, $s = \frac{1}{2}$ | $h_0 = const.,$ | Centre | $\frac{B_1 - 1}{\beta_1} > 1$ $\left(\frac{2B_1}{9} + \frac{1}{2\beta_1}\right)^2 > 1$ |

Table 2: Stability Analysis of the Fixed Point with MCG in TIS Theory

where $\Delta = \frac{4B_1^2}{9} - \frac{16}{9} + \frac{A_1}{4\beta_1 B_1^2} + \frac{1}{4\beta_1^2}$ and

$$\theta = \left(\frac{2B_1}{9} + \frac{1}{2\beta_1} + \frac{2\alpha(B_1 - \beta_1)}{9}\right)^2$$

IV. CONCLUSIONS

In this paper, we explore cosmological models of the universe in the presence of Modified Chaplygin gas (MCG) in addition to bulk viscosity type dissipative fluid which is described by Eckart or truncated causal theories proposed by Israel and Stewart. In Eckart theory, we note the following cosmological solutions: (i) for $s = \alpha = 0$, the scale factor grows exponentially, and the rate of expansion increases with increase in EoS parameter A and bulk viscosity parameter (β). (ii) for $s = \frac{1}{2}$, the scale factor grows exponentially and the rate of expansion is increases with decrease in α for a large value of A . (iii) de Sitter exponential expansion is obtained even in the presence of matter with $A = 0$, $s = \frac{1}{2}$ and $B = \sqrt{3}\beta - 1$. (iv) A singularity free power law evolution of the universe is obtained for $A = 0$, $s = \frac{1}{2}$ and $B = \sqrt{3}\beta - 1$. An accelerated ($q < 0$) power law expansion is obtained for following lower limit imposed on bulk viscosity by B as $\beta > \frac{3B+1}{3\sqrt{3}}$. In the case of TIS theory a Power law expansion is obtained for $s = \frac{1}{2}$ and $A = 0$. In this

case when $\beta = \frac{2\sqrt{3}}{11}$, it corresponds to an accelerating universe ($q = -0.5$) obtained even in a matter dominated era. This is a new solution which is due to viscosity considered in the universe filled with Chaplygin gas. In TIS theory exponential expansion with damped oscillatory behaviour is obtained for $s < \frac{1}{2}$ with $A = 0$ [54]. We note similar oscillatory behavior even in the presence of A . However the above mentioned behaviour may be obtain with nonlinear EoS ($A \neq 0$). Due inclusion of nonlinear term in the EoS i.e., for MCG the range of s is increased to obtain damped oscillator behaviour around an exponential inflation. The stability analysis of the fixed point is studied in both Eckart and TIS theory by linear approximation method. The fixed

TABLE 1

| Special case | Value of fixed point (h^*) | Type of fixed points | Evolution of scale factor $a(t)$ |
|-----------------------------|--|--|--|
| $s = 0, \alpha = 0$ | $\frac{\beta_1 + \sqrt{\beta_1^2 + 4A_1 B_1}}{2B_1}$ | Stable | $e^{\frac{\beta_1 + \sqrt{\beta_1^2 + 4A_1 B_1}}{2B_1} t}$ |
| | $\frac{\beta_1 - \sqrt{\beta_1^2 + 4A_1 B_1}}{2B_1}$ | Unstable | $e^{\frac{\beta_1 - \sqrt{\beta_1^2 + 4A_1 B_1}}{2B_1} t}$ |
| $s = \frac{1}{2}$ | $\left(\frac{A_1}{B_1 - \beta_1}\right)^{\frac{1}{2\alpha+2}}$ | Stable for $\beta_1 < B_1$ Unstable for $\beta_1 > B_1$ | $e^{\left(\frac{A_1}{B_1 - \beta_1}\right)^{\frac{1}{2\alpha+2}} t}$ |
| $A = 0, s \neq \frac{1}{2}$ | $\left(\frac{B_1}{\beta_1}\right)^{\frac{1}{2s-1}}$ | Stable for $s < \frac{1}{2}$, Unstable for $s > \frac{1}{2}$. | $e^{\left(\frac{B_1}{\beta_1}\right)^{\frac{1}{2s-1}} t}$ |
| $A = 0, s = \frac{1}{2}$ | $h_0 = const.,$ | Saddle Node bifurcation point for $B_1 = \beta_1$. | $e^{h_0 t}$ |

Table 1: Stability Analysis of the Fixed Point With MCG in Eckart Theory

TABLE 2

| Special case | Value of fixed point (h^*) | Type of fixed points | Condition for stability classification |
|---------------------|--|-------------------------------------|--|
| $s = 0, \alpha = 0$ | $\frac{\beta_1 + \sqrt{\beta_1^2 + 4A_1 B_1}}{2B_1}$ | Stable Node | $h^* > \frac{34A_1}{9\beta_1} - \frac{1}{4\beta_1 B_1^2} \Delta$ |
| | | Stable Spiral | $h^* < \frac{34A_1}{9\beta_1} - \frac{1}{4\beta_1 B_1^2} \Delta$ |
| $s = \frac{1}{2}$ | $\left(\frac{A_1}{B_1 - \beta_1}\right)^{\frac{1}{2\alpha+2}}$ | Stable Node | $\theta > \frac{4}{\beta_1} (1 + \alpha) \times (B_1 - \beta_1)$ |
| | | Stable Spiral | $\theta < \frac{4}{\beta_1} (1 + \alpha) \times (B_1 - \beta_1)$ |
| $A = 0$ | $\left(\frac{B_1}{\beta_1}\right)^{\frac{1}{2s-1}}$ | Stable Node for $s < \frac{1}{2}$, | $\left(\frac{2B_1}{9} + \frac{1}{2\beta_1}\right)^2 > 2(1-2s)$ |

points corresponding to exact solutions are tabulated in Table-1 and Table -2 in Eckart and TIS theory respectively. Here the fixed point represent a equilibrium solution which exhibits de Sitter type expansion ($a(t) \sim e^{h^*t}$) even in the presence of matter. A stable de Sitter type expansion (h^*) of the universe may be obtained both in Eckart and TIS theory for the following case: (1) $\alpha = 0$, $h^* = h^*_+$ (2) $s = \frac{1}{2}$, $\beta_1 < B_1$. These solution are particularly interesting because it can give rise to accelerated expansion and comparable with observational data [57]. The stable fixed points are further classified in TIS theory. If the rate of expansion of the de Sitter type evolution becomes higher the fixed points follow stable node type otherwise it will be stable spiral type when $i) s = 0 = \alpha, ii) s = \frac{1}{2}, A \neq 0$. For $s = \frac{1}{2}, A = 0$ the fixed point (h^*) is a saddle node bifurcation point in Eckart theory. In TIS theory a centre may be obtain for $s = \frac{1}{2}$ with $A = 0$. Thus cosmological models admitting expanding early expanding phase as well as late accelerating phase is possible in GR with exotic matter (MCG) and viscosity.

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